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Published in:
Ce/papers

Link to article, DOI:
10.1002/cepa.226

Publication date:
2017

Document Version
Peer reviewed version

A GBT-framework towards modal modelling of steel structures

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ABSTRACT

In modern structural steel frame design, the modelling of joints between beams and columns are based on very simple assumptions. The joints are most often assumed to behave as a perfect hinge or as a rigid joint. This means that in the overall static analysis relative rotations and changes in the moment curves due to joint deformations are neglected. This simplification eases the modelling but it is at the cost of losing a detailed understanding of the behaviour of the joint. This happens even though the European code has introduced the so-called component method in order to determine the rotational stiffness of a connection. Based on a modelling of any beam-to-column joint using finite shell elements and springs for single components such as bolts, it is the primary hypothesis that it is possible to formulate a generalized connection model with few degrees of freedom related to a relevant set of deformation modes. This hypothesis is based on the idea of modal decomposition performed in generalized beam theories (GBT). The question is – is it possible to formulate an eigenvalue problem with a solution corresponding to mode shapes for the deformation of the joint by using the finite element model and some type of GBT beam elements? It is believed that this is possible. The paper will address our investigations and show the progress of our research.

Keywords: Generalized beam theory, GBT, Connections, Thin-walled structures.

1 INTRODUCTION

In 1966 Schardt presented some of his research [1] based on the classic thin-walled beam theory developed by Vlasov [2] with some inspiration from plate theory in order to include in-plane distortion of the cross-section. Later, in 1972, Kollbrunner and Hajdin locked into the behaviour of folded open and closed thin-walled cross-sections [3] and in 1989 Schardt presented his formulation of the *Generalized Beam Theory* (GBT) (or in German: Verallgemeinerte Technische Biegetheorie – VTB) [4]. In the last decades researchers have put an effort into the field of GBT to extend and modify this theory e.g. see Davies & Leach [5], Camotim & Silvestre [6], Jönsson & Andreassen [7] and Miranda *et.al.* [8].

The principle in GBT, i.e. Generalized Beam Theory, as presented by Schardt is to formulate differential equilibrium equations similar to those used in traditional beam theory (e.g. see [9]). These differential equilibrium equations are then solved based on natural mode shapes that are determined by minimizing the energy and solving a corresponding eigenvalue problem. By choosing, the mode shapes based on the energy content and relevance it is possible to set up a very efficient GBT beam element. Based on a modelling of any beam-to-column joint using finite shell elements and springs for single components such as bolts is it the primary hypothesis that it is possible to formulate a generalized relatively simple connection model with few degrees of freedom related to a relevant set of deformation modes. This hypothesis is based on the idea of modal decomposition performed in generalized beam theories as done in [10].

This paper will introduce the establishment of the cross-section displacement formulations for a GBT beam element. This beam element will be the foundation for a GBT-framework leading to the investigation of beam-to-column joints. Throughout section 2 and 3 some of the essential assumptions for this GBT formulation will be established. This will include the basic kinematic assumptions

concerning assumed displacement fields and strain energy. To describe different displacement modes the cross-section is discretized into straight elements. Section 4 goes into the interpolation of these "wall" elements within the cross-section. In section 5, the differential equilibrium equations are established through virtual variation of the potential energy formulation. These higher ordered differential equations will be solved by reducing the order and introducing a state vector. The resulting linear differential equations of first order are solved by seeking solutions of polynomial and exponential format, through the solution of the related eigenvalue problem. Finally, section 6 and 7 presents a perspective of the ideas and a summary of the content in this paper.

2 KINEMATIC ASSUMPTIONS

A prismatic thin-walled member is located in a global Cartesian coordinate system (*X, Y, Z*) where the *Z*-axis is pointing in the longitudinal direction of the member. To navigate through the cross-section located in the *XY*-plain of the member a local coordinate system is introduced as (*n, s, z*). Here *s* is a curve parameter running along the centre line of the cross-section and *n* indicates the normal to *s*. The local axial direction *z* is parallel to the global *Z*-axis. In *Fig. 1* both the local (*n, s, z*) and global (*X, Y, Z*) coordinate systems are indicated.

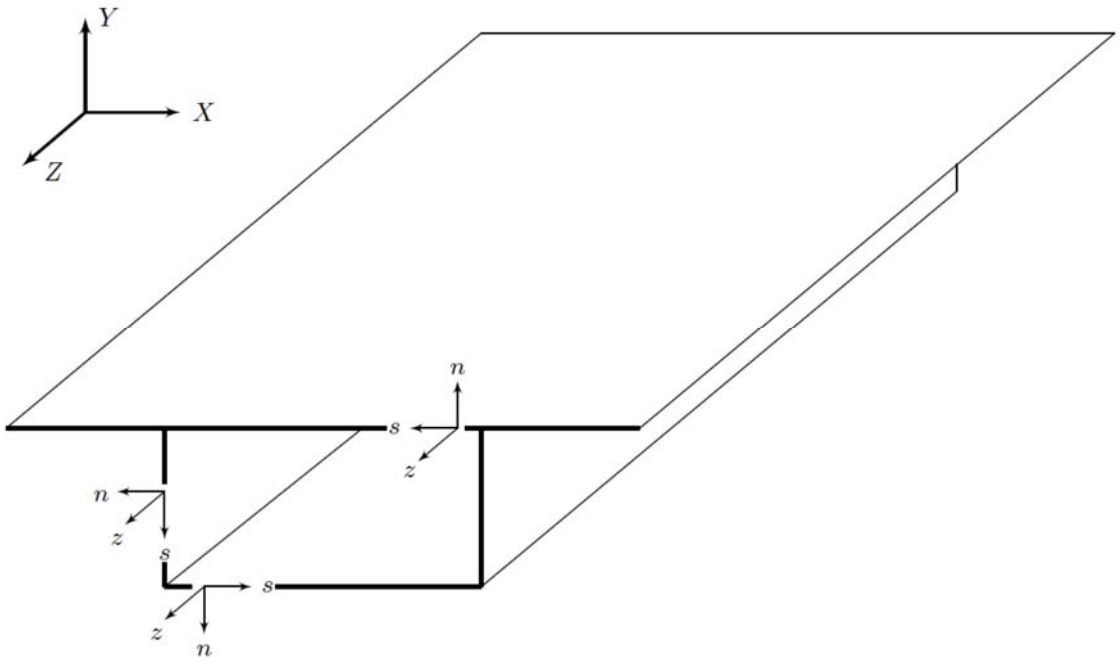


Fig. 1. A thin-walled structure located in a global Cartesian coordinate system (*X, Y, Z*). The local coordinate systems (*n, s, z*) are indicated as well

In this paper subscript *n* and *s* refer to components in local coordinates corresponding to normal and tangential directions. Derivatives will be written as a subscript following a comma such as $(\cdot)_{,n} \equiv d(\cdot)/dn$ or $(\cdot)_{,ss} \equiv d^2(\cdot)/ds^2$ except axial variation which will be noted by a prime, $(\cdot)' \equiv d(\cdot)/dz$.

In classic beam theory [9] we normally decouple displacements into: extension, flexure, torsion, warping and distortion. Each of these modes contributes to the elastic energy formulation and taking the sum of all these yields the total contribution. However, classical thin-walled beam theory [2] differentiate the displacement fields into three main directions following the global coordinate axes. Moreover, the theory presented here includes membrane shear strain as well as transverse extension along the cross-section mid-line, but disregards the Poisson effect. The influence of curved cross-section walls is neglected in the strain formulations and wall elements will be assumed straight. The theory presented includes local shear deformation effects in the axial direction. Following the

principles of thin-walled beam theory the in-plane displacements of an arbitrary point (n, s) in the cross-section can be described as:

$$u_n(s,z) = w_n(s)\,\psi(z) \tag{1}$$

$$u_s(n,s,z) = \left(w_s(s) - n\,w_{n,s}(s) \right)\psi(z) \tag{2}$$

where $w_n(s)$ and $w_s(s)$ are the centre line displacements in local directions factorized by a function of the axial coordinate expressed as $\psi(z)$. Regarding out-of-plane the displacement field can be written as:

$$u_z(n,s,z) = (\,\Omega(s) + n\,\alpha(s)\,)\,\eta(z) \tag{3}$$

where $\Omega(s)$ is the axial displacement mode (warping) for the centre line of the wall, $\alpha(s)$ describes the inclination through the thickness, and $\eta(z)$ is the function that describes the axial variation.

For simplicity the function parentheses such as (s) in *Eq. (1)* to *(3)* will be left out in the following. *Fig. 2* shows the displacements. It is assumed that the displacements and rotations are small.

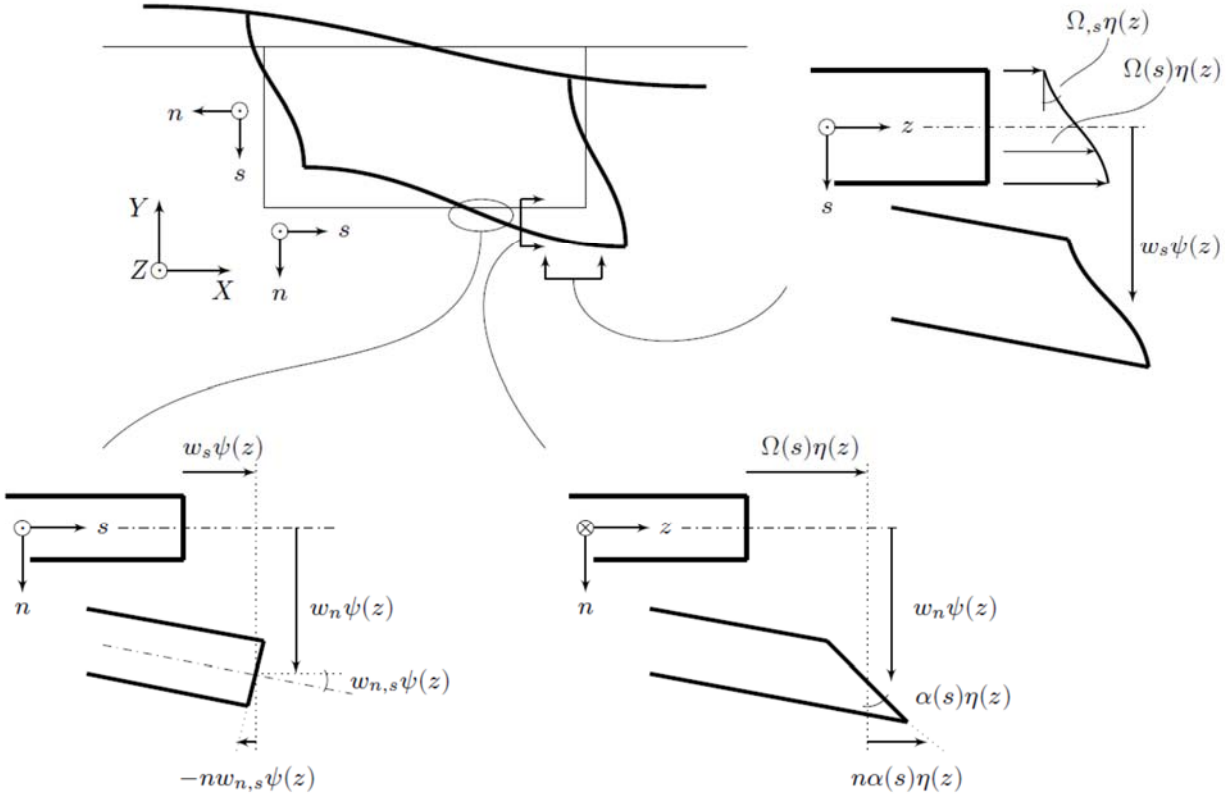


Fig. 2. Cross-section with illustrations of displacement fields

Having the kinematic relations from *Eq. (1)* to *(3)* we can now find the strains in a cross-section. Using linear strain definitions and using engineering shear strains, the following strains are found:

$$\varepsilon_{ss} = u_{s,s} = \left(w_{s,s} - n\,w_{n,ss}\right)\psi \tag{4}$$

$$\varepsilon_{zz} = u_z{}' = (\Omega + n\,\alpha)\,\eta' \tag{5}$$

$$\gamma_{nz} = 2\varepsilon_{nz} = u_n{}' + u_{z,n} = w_n\,\psi' + \alpha\,\eta \tag{6}$$

$$\gamma_{sz} = 2\varepsilon_{sz} = u_s{}' + u_{z,s} = \left(w_s - n\,w_{n,s}\right)\psi' + \left(\Omega_{,s} + n\,\alpha_{,s}\right)\eta \tag{7}$$

For the remaining strains we have: $\varepsilon_{nn} = \gamma_{ns} = \gamma_{sn} = 0$, $\gamma_{zn} = \gamma_{nz}$, and $\gamma_{zs} = \gamma_{sz}$.

As shown in *Fig. 2* and from *Eq. (6)* and *(7)* it can be seen that shear strains are included in the theory. The inclusion of shear deformations and shear strains will allow the modelling of torsion in closed cells as well as shear deformations in short beams. Miranda *et.al.* [8] utilizes some of the same ideas including shear strains in their work describing a thin-walled beam member.

3 STRAIN ENERGY ASSUMPTIONS

This section deals with the formulation of the elastic potential energy using simple constitutive relations. The material is assumed to be linear elastic with a modulus of elasticity E and a shear modulus $G = E/(2(1+\nu))$, in which ν represents the Poisson ratio. Furthermore a plate type elasticity modulus is used in the transverse direction $E_s = E/(1-\nu^2)$. The axial stress is $\sigma = E\varepsilon$, the shear stress is $\tau = G\gamma$, and the transverse stress is assumed to be given by $\sigma_s = E_s\varepsilon_s$.

The constitutive equations in the form of a generalized Hooke's law in which the Poisson effect is neglected can be written as:

$$\begin{bmatrix} \sigma_{zz} \\ \sigma_{ss} \\ \tau_{nz} \\ \tau_{sz} \end{bmatrix} = \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E_s & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & G \end{bmatrix} \begin{bmatrix} \varepsilon_{zz} \\ \varepsilon_{ss} \\ \gamma_{nz} \\ \gamma_{sz} \end{bmatrix} \tag{8}$$

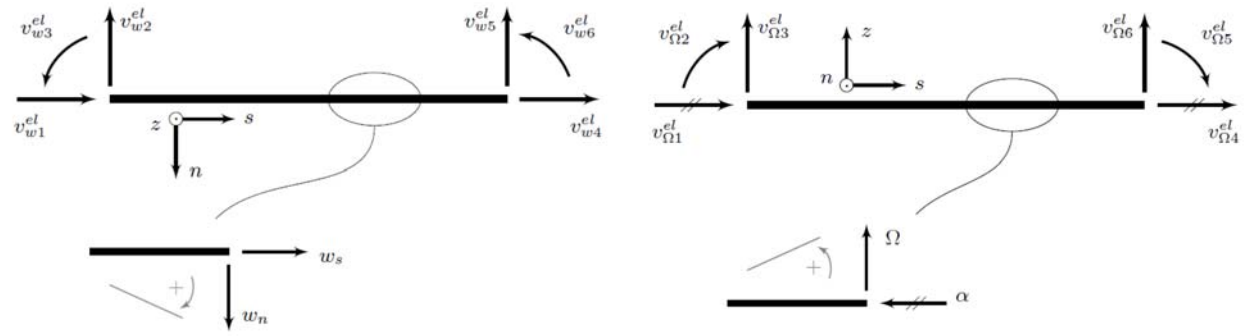
With these constitutive relations the elastic potential energy of the system can be written as:

$$\Pi = \frac{1}{2}\int_V \left(E_s\varepsilon_{ss}^2 + E\varepsilon_{zz}^2 + G\gamma_{nz}^2 + G\gamma_{sz}^2 \right) dV \tag{9}$$

Given the potential energy in *Eq. (9)* we can substitute the strains found in *Eq. (4)* to *(7)* based on the assumed displacement fields. Demanding a minimized solution to this equation leads to a set of differential equations – these are the equilibrium equations. In the following sections we will derive these equations by discretize the cross-section into smaller straight "wall" elements. For each of these elements we introduce interpolation functions in order to be able to describe the displacements.

4 INTERPOLATION WITHIN CROSS-SECTION ELEMENTS

As mentioned above let us consider a prismatic thin-walled cross-section assembled by straight beam elements such as the one shown in *Fig. 3*. The figure illustrates an element and its degrees of freedom (dof). The left part shows the dof regarding in-plane movements and the right part of the figure illustrates the out-of-plane dof. The displacements are interpolated using these dof. The displacements w_s and the "warping" rotations α are interpolated linearly, whereas the displacement w_n and warping



a) b)

Fig. 3. Beam element illustrating degrees of freedom used for cross-section discretization
a) in-plane degrees of freedom b) out-of-plane degrees of freedom.
The cut-outs illustrates the sign-convention used for the displacements

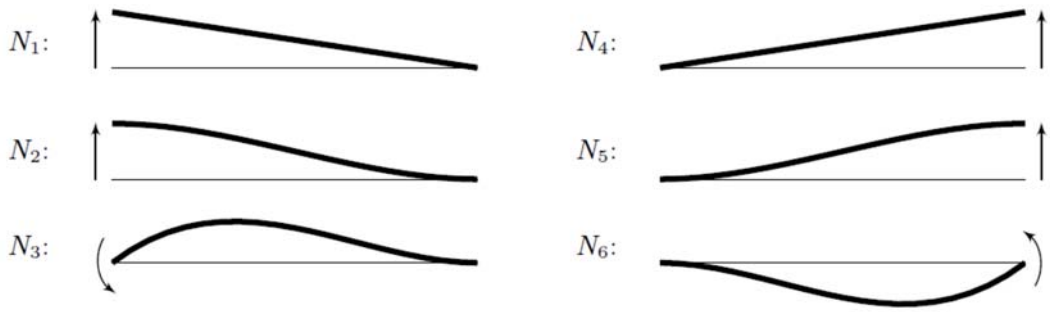


Fig. 4. Shape functions, from *Eq. (10)*. The arrows indicate a unit displacement or rotation for either the left node (the column to the left) or the right node (the column to the right), respectively. A displacement above the horizontal line yields positive values

displacement Ω are interpolated with cubic functions. *Fig. 4* illustrates the variation of the interpolation functions.

The interpolation functions used are given by:

$$\begin{aligned} N_1 &= 1-\xi \quad , \quad N_4 = \xi \\ N_2 &= (1+2\xi)(1-\xi)^2 \quad , \quad N_5 = \xi^2(3-2\xi) \\ N_3 &= \xi(1-\xi)^2 b_{el} \quad , \quad N_6 = -\xi^2(1-\xi)b_{el} \end{aligned} \tag{10}$$

where ξ is the normalized coordinate along the element, given as: $\xi = s/b_{el}$
b_{el} is the length of a single discretized element

For a single element we collect the dof in two vectors; one representing in-plane movements and the other representing out-of-plane movements by:

$$\mathbf{v}_w^{el} = [v_{w1}^{el} \quad v_{w2}^{el} \quad v_{w3}^{el} \quad v_{w4}^{el} \quad v_{w5}^{el} \quad v_{w6}^{el}]^T \tag{11}$$

$$\mathbf{v}_\Omega^{el} = [v_{\Omega1}^{el} \quad v_{\Omega2}^{el} \quad v_{\Omega3}^{el} \quad v_{\Omega4}^{el} \quad v_{\Omega5}^{el} \quad v_{\Omega6}^{el}]^T \tag{12}$$

The internal displacements of an element in the local coordinate system are thus given by interpolation functions as:

$$w_s\,\psi = \mathbf{N}_s\,\mathbf{v}_w^{el}\,\psi \quad , \quad \mathbf{N}_s = [N_1 \quad 0 \quad 0 \quad N_4 \quad 0 \quad 0] \tag{13}$$

$$w_n\,\psi = \mathbf{N}_n\,\mathbf{v}_w^{el}\,\psi \quad , \quad \mathbf{N}_n = [0 \quad -N_2 \quad -N_3 \quad 0 \quad -N_5 \quad -N_6] \tag{14}$$

$$\alpha\,\eta = \mathbf{N}_\alpha\,\mathbf{v}_\Omega^{el}\,\eta \quad , \quad \mathbf{N}_\alpha = [-N_1 \quad 0 \quad 0 \quad -N_4 \quad 0 \quad 0] \tag{15}$$

$$\Omega\,\eta = \mathbf{N}_\Omega\,\mathbf{v}_\Omega^{el}\,\eta \quad , \quad \mathbf{N}_\Omega = [0 \quad -N_3 \quad N_2 \quad 0 \quad -N_6 \quad N_5] \tag{16}$$

where $\mathbf{N}_s$ and $\mathbf{N}_\alpha$ are linear interpolation matrices whereas $\mathbf{N}_n$ and $\mathbf{N}_\Omega$ are cubic (beam) interpolation matrices. The signs are introduced due to the special sign in the (*n*, *s*) system.

The cross-section is discretized into several "wall" elements and the total potential energy may be found by summing the element contributions. The integration of the potential energy *Eq. (9)* is split into three integrations respectively over the thickness, the width and the length. The integration through the thickness is simplified using;

$$\int_{-t/2}^{t/2} C\ dn = tC \quad , \quad \int_{-t/2}^{t/2} nC\ dn = 0 \quad , \quad \int_{-t/2}^{t/2} n^2 C\ dn = \frac{1}{12}t^3 C \tag{17}$$

where *C* is a constant, *t* is the thickness of the element, and *n* is the transverse coordinate axes.

Introducing the strains from *Eq. (4)* to *(7)*, and integrating over the thickness using *Eq. (17)* results in the following total potential energy:

$$\begin{aligned}\Pi = \frac{1}{2}\int_0^l \sum_{\text{el}} \int_0^{b_{el}} \Big\{ & E_s\left(t\left(w_{s,s}\,\psi\right)^2 + \frac{1}{12}t^3\left(w_{n,ss}\,\psi\right)^2\right) \\ & + E\left(t(\Omega\,\eta')^2 + \frac{1}{12}t^3(\alpha\,\eta')^2\right) \\ & + G(t(\alpha\,\eta)^2 + t(w_n\,\psi')^2 + 2t\,\alpha\,\eta\,w_n\,\psi') \\ & + G\left(t\left(\Omega_{,s}\,\eta\right)^2 + \frac{1}{12}t^3\left(\alpha_{,s}\,\eta\right)^2 + t(w_s\,\psi')^2 + \frac{1}{12}t^3\left(w_{n,s}\,\psi'\right)^2\right. \\ & \left. + \,2t\,\Omega_{,s}\,\eta\,w_s\,\psi' - 2\frac{1}{12}t^3\,\alpha_{,s}\,\eta\,w_{n,s}\,\psi'\right)\Big\}\,ds\,dz\end{aligned} \tag{18}$$

We can now substitute the interpolation functions of *Eq. (13)* to *(16)* into *Eq. (18)* and sort the stiffness terms in ψ and η yielding the local stiffness matrices given in *Table 1*.

Table 1. Local stiffness matrices for a single discretized element

$$\mathbf{k}^s = \int_0^{b_{el}} \left(E_s t\,\mathbf{N}_{s,s}^T\mathbf{N}_{s,s} + \frac{1}{12}E_s t^3\mathbf{N}_{n,ss}^T\mathbf{N}_{n,ss}\right) ds$$

$$\mathbf{k}_{\Omega\Omega}^{\sigma} = \int_0^{b_{el}} \left(E\,t\,\mathbf{N}_{\Omega}^T\mathbf{N}_{\Omega} + \frac{1}{12}E\,t^3\mathbf{N}_{\alpha}^T\mathbf{N}_{\alpha}\right)\,ds$$

$$\mathbf{k}_{\Omega\Omega}^{\gamma} = \int_0^{b_{el}} \left(G\,t\,\mathbf{N}_{\alpha}^T\mathbf{N}_{\alpha} + G\,t\,\mathbf{N}_{\Omega,s}^T\mathbf{N}_{\Omega,s} + \frac{1}{12}G\,t^3\mathbf{N}_{\alpha,s}^T\mathbf{N}_{\alpha,s}\right) ds$$

$$\mathbf{k}_{ww}^{\gamma} = \int_0^{b_{el}} \left(G\,t\,\mathbf{N}_{\mathrm{n}}^T\mathbf{N}_n + G\,t\,\mathbf{N}_s^T\mathbf{N}_s + \frac{1}{12}G\,t^3\mathbf{N}_{n,s}^T\mathbf{N}_{n,s}\right) ds$$

$$\mathbf{k}_{\Omega\mathrm{w}}^{\gamma} = \int_0^{b_{el}} \left(G\,t\,\mathbf{N}_{\alpha}^T\mathbf{N}_n + G\,t\,\mathbf{N}_{\Omega,s}^T\mathbf{N}_s - \frac{1}{12}G\,t^3\mathbf{N}_{\alpha,s}^T\mathbf{N}_{n,s}\right) ds$$

$$\mathbf{k}_{w\Omega}^{\gamma} = \int_0^{b_{el}} \left(G\,t\,\mathbf{N}_n^T\mathbf{N}_{\alpha} + G\,t\,\mathbf{N}_s^T\mathbf{N}_{\Omega,s} - \frac{1}{12}G\,t^3\mathbf{N}_{n,s}^T\mathbf{N}_{\alpha,s}\right) ds$$

Using the notation of the local stiffness matrices in *Table 1* the elastic potential energy can be written as:

$$\begin{aligned}\Pi = \frac{1}{2}\int_0^l \sum_{\text{el}} \Big\{ & \psi\,\mathbf{v}_w^{el^T}\,\mathbf{k}^s\,\mathbf{v}_w^{el}\psi \;+\; \eta'\mathbf{v}_{\Omega}^{el^T}\,\mathbf{k}_{\Omega\Omega}^{\sigma}\,\mathbf{v}_{\Omega}^{el}\eta' \;+\; \eta\,\mathbf{v}_{\Omega}^{el^T}\,\mathbf{k}_{\Omega\Omega}^{\gamma}\,\mathbf{v}_{\Omega}^{el}\,\eta \\ & + \psi'\mathbf{v}_w^{el^T}\mathbf{k}_{ww}^{\gamma}\mathbf{v}_w^{el}\psi' \;+\; \eta\,\mathbf{v}_{\Omega}^{el^T}\mathbf{k}_{\Omega\mathrm{w}}^{\gamma}\,\mathbf{v}_{\mathrm{w}}^{el}\psi' \;+ \\ & \psi'\mathbf{v}_{\mathrm{w}}^{el^T}\mathbf{k}_{w\Omega}^{\gamma}\,\mathbf{v}_{\Omega}^{el}\eta\,\Big\}\,dz\end{aligned} \tag{19}$$

By assembling all the elements representing the discretized cross-section, computing the local stiffness matrices, *Table 1*, and transforming from local dof to global dof the entire cross-section energy is formulated. A standard transformation matrix is used to rotate dof from $\mathbf{v}_w^{el}$ and $\mathbf{v}_{\Omega}^{el}$ in local coordinates into global coordinates as $\mathbf{V}_w$ and $\mathbf{V}_{\Omega}$. The global format of the potential energy can thus be written as follows:

$$\begin{aligned}\Pi = \frac{1}{2}\int_0^l \{ & \psi\,\mathbf{V}_w^T\,\mathbf{K}^s\,\mathbf{V}_w\psi \;+\; \eta'\mathbf{V}_{\Omega}^T\,\mathbf{K}_{\Omega\Omega}^{\sigma}\,\mathbf{V}_{\Omega}\eta' \;+\; \eta\,\mathbf{V}_{\Omega}^T\,\mathbf{K}_{\Omega\Omega}^{\gamma}\,\mathbf{V}_{\Omega}\,\eta \\ & + \psi'\mathbf{V}_w^T\mathbf{K}_{ww}^{\gamma}\mathbf{V}_w\psi' \;+\; \eta\,\mathbf{V}_{\Omega}^T\mathbf{K}_{\Omega\mathrm{w}}^{\gamma}\,\mathbf{V}_w\psi' \;+\; \psi'\mathbf{V}_w^T\mathbf{K}_{w\Omega}^{\gamma}\,\mathbf{V}_{\Omega}\eta\}\,dz\end{aligned} \tag{20}$$

In which a bold capital **K** indicates that it is the global stiffness matrix for the entire cross-section.

5 THE DIFFERENTIAL EQUATIONS

This section deals with setting up and solving the differential equilibrium equations. First, the differential equilibrium equations are determined by performing a virtual variation of the energy formulation from *Eq. (20)*. Secondly, the order will be reduced. This reduction of order is performed as done in [11] and [12]. Finally, this section also presents the solutions to the homogeneous differential equations consisting of exponential solutions and polynomial solutions.

5.1 Establishing the differential equilibrium equations

Working with potential energy it is possible to determine equilibrium equations and static boundary conditions by minimization and thus demanding stationarity of the potential energy. In other words the first virtual variation of the potential energy is required to be equal zero in order to minimize it. If displacement variations $\delta\psi$ and $\delta\eta$ satisfies the kinematic boundary conditions the variation $\delta\Pi$ must vanish. Therefor we perform a partial integration of some of the terms from *Eq. (20)* in order to be able to have all terms depending on the basic virtual variations $\delta\psi$ and $\delta\eta$. The first variation of the elastic potential energy can be written in a convenient matrix notation as:

$$\begin{aligned}\delta\Pi = \int_0^l \delta[\psi\mathbf{V}_w^T \quad \eta\mathbf{V}_\Omega^T] \Bigg(\begin{bmatrix} -\mathbf{K}_{ww}^{\gamma} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K}_{\Omega\Omega}^{\sigma} \end{bmatrix} \begin{bmatrix} \psi\mathbf{V}_w \\ \eta\mathbf{V}_\Omega \end{bmatrix}'' \\ + \begin{bmatrix} \mathbf{0} & -\mathbf{K}_{w\Omega}^{\gamma} \\ \mathbf{K}_{\Omega w}^{\gamma} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \psi\mathbf{V}_w \\ \eta\mathbf{V}_\Omega \end{bmatrix}' + \\ \begin{bmatrix} \mathbf{K}^{s} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{\Omega\Omega}^{\gamma} \end{bmatrix} \begin{bmatrix} \psi\mathbf{V}_w \\ \eta\mathbf{V}_\Omega \end{bmatrix} \Bigg) dz \\ + \left[\delta[\psi\mathbf{v}_w^T \quad \eta\mathbf{v}_\Omega^T] \left(\begin{bmatrix} \mathbf{K}_{ww}^{\gamma} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{\Omega\Omega}^{\sigma} \end{bmatrix} \begin{bmatrix} \psi\mathbf{V}_w \\ \eta\mathbf{V}_\Omega \end{bmatrix}' + \begin{bmatrix} \mathbf{0} & \mathbf{K}_{w\Omega}^{\gamma} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \psi\mathbf{V}_w \\ \eta\mathbf{V}_\Omega \end{bmatrix} \right) \right]_0^l \end{aligned} \tag{21}$$

Here a bold **0** denotes a suitable vector or matrix of zeros. This notation will be used throughout this paper. The last term in the squared brackets is a result of the partial integration and corresponds to the boundary conditions and boundary loads.

From *Eq. (21)* we can extract the equilibrium equations and demanding them to be zero as done in *Eq. (22)*. To obtain symmetry in the matrices the lower row-equation-set is multiplied by -1.

$$\overbrace{\begin{bmatrix} -\mathbf{K}_{ww}^{\gamma} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{\Omega\Omega}^{\sigma} \end{bmatrix}}^{\mathbf{K_2}} \begin{bmatrix} \psi\mathbf{V}_w \\ \eta\mathbf{V}_\Omega \end{bmatrix}'' + \overbrace{\begin{bmatrix} \mathbf{0} & -\mathbf{K}_{w\Omega}^{\gamma} \\ -\mathbf{K}_{\Omega w}^{\gamma} & \mathbf{0} \end{bmatrix}}^{\mathbf{K_1}} \begin{bmatrix} \psi\mathbf{V}_w \\ \eta\mathbf{V}_\Omega \end{bmatrix}' + \overbrace{\begin{bmatrix} \mathbf{K}^{s} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K}_{\Omega\Omega}^{\gamma} \end{bmatrix}}^{\mathbf{K_0}} \begin{bmatrix} \psi\mathbf{V}_w \\ \eta\mathbf{V}_\Omega \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{22}$$

It is interesting to note that $\mathbf{K}^s$ is three times singular, due to the three in-plane rigid body movements of the cross-section (displacements in the *X*- and *Y*-axis and a rotation along the *Z*-axis). These are non-distortional displacements. The shear stiffness matrix $\mathbf{K}_{\Omega\Omega}^{\gamma}$ has one singularity, related to an axial rigid body movement of the entire section. We can write *Eq. (22)* in a simpler form similar to the differential equations used in traditional GBT.

$$\mathbf{K_2}\mathbf{V}'' + \mathbf{K_1}\mathbf{V}' + \mathbf{K_0}\mathbf{V} = \mathbf{0} \tag{23}$$

where $\mathbf{K_2}$, $\mathbf{K_1}$, $\mathbf{K_0}$ represent overall stiffness matrices as indicated above (see *Eq. (22)*),
$\mathbf{V}$ is the displacement vector given as: $\mathbf{V} = [\psi\mathbf{V}_w \quad \eta\mathbf{V}_\Omega]^T$

In contrast to the traditional GBT formulation it is seen from *Eq. (23)* that it is only a second ordered differential equation, whereas Schardt and Davies & Leach presents fourth ordered differential equations [4, 5].

5.2 Reduction of order

Introducing a new variable it is possible to reduce the higher ordered differential equation system described by *Eq. (23)* into a first order differential equation system. Such a system has a linear matrix eigenvalue problem that can easily be solved. With this operation the number of unknowns is unfortunately increased. In our case a first order reduction is enough. We introduce the new variable $\mathbf{Y} = \mathbf{V}'$ and substitute this into *Eq. (23)*, which leads to:

$$\begin{aligned} &\mathbf{Y} - \mathbf{V}' = \mathbf{0} \\ &\mathbf{K}_2\mathbf{Y}' + \mathbf{K}_1\mathbf{Y} + \mathbf{K}_0\mathbf{V} = \mathbf{0} \end{aligned} \tag{24}$$

Thereby we have reduced the order, but we have also doubled the number of unknowns. If we collect the two variables in a state vector as $\overline{\mathbf{V}} = [\mathbf{V} \quad \mathbf{Y}]^T$ we can write the more convenient matrix equation system of double size:

$$\begin{bmatrix} \mathbf{K_0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K_2} \end{bmatrix} \overline{\mathbf{V}} + \begin{bmatrix} \mathbf{K_1} & \mathbf{K_2} \\ \mathbf{K_2} & \mathbf{0} \end{bmatrix} \overline{\mathbf{V}}' = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{25}$$

where the first row corresponds to the original equation from *Eq. (23)* and the second row represents a trivial identity equation also given in the upper formulation of *Eq. (24)*. Substituting $\overline{\mathbf{V}}$ we have:

$$\begin{bmatrix} \mathbf{K_0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K_2} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{V}' \end{bmatrix} + \begin{bmatrix} \mathbf{K_1} & \mathbf{K_2} \\ \mathbf{K_2} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{V}' \end{bmatrix}' = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{26}$$

It is known that $\begin{bmatrix} \mathbf{K_0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K_2} \end{bmatrix}$ is four times singular whereas $\begin{bmatrix} \mathbf{K_1} & \mathbf{K_2} \\ \mathbf{K_2} & \mathbf{0} \end{bmatrix}$ has full rank.

5.3 Solutions to differential equations

The solutions to the differential equations, *Eq. (26)*, consist of two parts; exponential solutions and polynomial solutions.

Seeking solutions of exponential form, $\psi(z) = e^{\lambda z}$ and $\eta(z) = e^{\lambda z}$ leads to a generalized linear eigenvalue problem:

$$\begin{bmatrix} \mathbf{K_0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K_2} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{Y} \end{bmatrix} + \lambda \begin{bmatrix} \mathbf{K_1} & \mathbf{K_2} \\ \mathbf{K_2} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{Y} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{27}$$

where λ holds the eigenvalues and $\mathbf{V}$ and $\mathbf{Y}$ represents the transverse mode shape and the change of mode shape along the beam, respectively. The modes representing zero eigenvalues correspond to polynomial solutions with an order dependent on the degree of the zero roots. These modes will have to be clearly identified or separated into the conventional beam modes.

6 PERSPECTIVES

The new ideas indicated in this paper regarding GBT will be used as the foundation for a similar method for joints where the boundary conditions for the adjacent GBT beam elements are used to create an internal model for the joint. The boundary conditions for a GBT beam element can be formulated so that they are similar to the boundary conditions of shell elements (see [11]). On this basis, the natural deformation mode shapes will be determined based on the related eigenvalue problem and the relevant reduced set of mode shapes are then chosen by the content of energy, relevance, and boundary conditions. To ensure that the method will be applicable it is the intention to develop and introduce a number of special 3D component elements for bolts, plates with local movements, contact and pre-tension.

The proposed method opens for new possibilities regarding the overall static computations of steel structures. The influence of relative movements and rotations at the beam-ends will be taken into account.

Energy principles and variation methods are used to model beams and connections formulated in matrix and vector format. Variations of the energy formulation results in equilibrium equations, which are solved using approximated shape functions or semi-exact shape functions based on simplified assumptions.

7 SUMMARY AND ACKNOWLEDGMENT

This paper presents the idea and base of a GBT-framework investigating thin-walled structures. The formulation is a branching of the classical GBT formulation given by Schardt in 1989 [4] relying on Vlasovs classical thin-walled theory [2]. The framework, presented here, is designed to be extended in order to be able to model and analyse beam-to-column joints.

For a prismatic thin-walled cross-section a set of displacement fields are established in order to include shear deformations of the cross-section. From the kinematic relations strains and engineering shear strains are determined, which among others allow modelling of torsion in closed cells and short beams. Furthermore, the theory presented includes membrane shear strain as well as transverse extension along the cross-section mid-line, and local shear deformation effects in the axial direction. Using constitutive relations neglecting the Poisson effect an elastic potential energy equation is formulated. To decouple the cross-section deformation modes the cross-section is discretized into straight beam elements from which the displacements are computed using interpolation functions. Demanding a minimized solution of a virtual variation of the elastic potential energy equation leads to a set of differential equilibrium equations. In conclusion it is discussed how the system of differential equilibrium equations can be solved using an eigenvalue problem and polynomial solutions.

The first author gratefully acknowledges financial support by the Danish engineering consultancy ALECTIA A/S and the Innovation Fund Denmark, grant: 5189-00005B.

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